PHYSICAL REVIEW E **72**, 020101(R) (2005)

Exact solution of the Fokker-Planck equation for a broad class of diffusion coefficients

Kwok Sau Fa

Departamento de Física, Universidade Estadual de Maringá, Avenida Colombo, 5790, 87020-900, Maringá-Paraná, Brazil

(Received 15 March 2005; revised manuscript received 31 May 2005; published 18 August 2005)

We consider the Langevin equation with a multiplicative noise term that depends on time and space. The corresponding Fokker-Planck equation in the Stratonovich approach is investigated. Its exact solution is obtained for an arbitrary multiplicative noise term given by $g(x,t)=D(x)T(t)$, and the behaviors of probability distributions, for some specific functions of $D(x)$, are analyzed. We show that the asymptotic shape of the random-walk model and power-law decay obtained from other approaches can be reproduced from our solutions, by employing two simple functions for $g(x,t)$. In particular, for $D(x)\sim|x|^{-\theta/2}$, the physical solutions for the probability distribution in the Ito, Stratonovich, and postpoint discretization approaches can be obtained and analyzed.

DOI: 10.1103/PhysRevE.72.020101 PACS number(s): 05.40.−a, 05.60.−k, 66.10.Cb

In the last several decades, anomalous diffusion properties have been extensively investigated by several approaches in order to model different kinds of probability distributions such as long-range spatial or temporal correlations [1]. For instance, the well-known cases are the Langevin and the corresponding Fokker-Planck equations, and the master equation. The other ones we could mention are the generalized Langevin equations [2], the generalized Fokker-Planck equation with memory effect [3], generalized thermostatistics [4], generalized master equations [5], continuous time random walk [6], and fractional equations [1]. These approaches have been used to describe numerous systems in several contexts such as physics, hydrology, chemistry, and biology.

The well-established property of the normal diffusion described by the Gaussian distribution can be obtained by the usual Fokker-Planck equation with a constant diffusion coefficient (without the drift term). Anomalous diffusion regimes can also be obtained by the usual Fokker-Planck equation; however, they arise from a variable diffusion coefficient that depends on time and/or space. On the other hand, in the view of Langevin approach, it is associated with a multiplicative noise term. In other approaches such as the generalized Fokker-Planck equation (nonlinear) and fractional equations, they can describe anomalous diffusion regimes with a constant diffusion coefficient.

In this paper, we investigate the Fokker-Planck equation with a variable diffusion coefficient in time and space, in the Stratonovich approach. We show that for a multiplicative noise term separable in time and space, $g(x,t)=D(x)T(t)$, we can obtain a formal solution for the probability distribution. We also analyze the behaviors of probability distributions for some specific functions of $D(x)$, which can manifest interesting properties such as non-Gaussian distribution, combination of behaviors such as Gaussian (for small distances) and exponential (for large distances), and combination of behaviors such as Gaussian (for small distances) and power-law decay for long distances. Also, we can obtain many bimodal distributions for different forms of $D(x)$.

Now, we consider the following Langevin equation:

$$\dot{\xi}=g(\xi,t)\Gamma(t), \tag{1}$$

where ξ is a stochastic variable and $\Gamma(t)$ is the Langevin force. We assume that the averages $\langle\Gamma(t)\rangle=0$ and $\langle\Gamma(t)\Gamma(\bar{t})\rangle=2\delta(t-\bar{t})$ [3]. For $g=\sqrt{D}$, Eq. (1) describes the Wiener process, and the corresponding probability distribution is described by a Gaussian function. In the case of the $g(\xi,t)$ variable, some specific functions have been employed to study, for instance, turbulent flows ($g(x,t)\sim|x|^a t^b$) [7,8]. By applying the Stratonovich approach in a one-dimensional space [3], we obtain the following dynamical equation for the probability distribution:

$$\frac{\partial W(x,t)}{\partial t}=\frac{\partial}{\partial x}\left[g(x,t)\frac{\partial[g(x,t)W(x,t)]}{\partial x}\right]. \tag{2}$$

Hereafter we consider that the multiplicative noise term $g(x,t)$ is separable in time and space, $g(x,t)=D(x)T(t)$. Then, we can rewrite Eq. (2) in the following manner:

$$\frac{\partial\rho}{\partial\bar{t}}=\frac{\partial^2\rho}{\partial\bar{x}^2}, \tag{3}$$

with

$$\rho(x,t)=D(x)W(x,t), \tag{4}$$

$$\frac{d\bar{t}}{dt}=T^2(t), \tag{5}$$

and

$$\frac{d\bar{x}}{dx}=\frac{1}{D(x)}. \tag{6}$$

Equation (3) has the following formal solution,

$$\rho(\bar{x},\bar{t})=A\frac{\exp\left[-\frac{\bar{x}^2}{4\bar{t}}\right]}{\sqrt{\bar{t}}}, \tag{7}$$

where A is a normalization factor. We note that for $D(x)=\sqrt{D}$ and $T(t)=1$ we recover the Wiener process.

TABLE I. Asymptotic shapes of the solutions (9) and (14).

Solution (9)	Asymptotic shape	Solution (14)	Asymptotic shape
$a=0$ or $b=0$	Gaussian	$\theta=0$	Gaussian
$c=1/b$	Power law	$\theta>0$	Compressed Gaussian
$b=1$, $c=1/2$	Exponential	$-2<\theta<0$	Stretched Gaussian
$bc<0$	Compressed Gaussian		
$bc>0$	Stretched Gaussian		

We can now explore some spatial features of the probability distribution of solution (4), for some specific forms of $D(x)$.

First case. We consider

$$\bar{x} = \frac{x}{\sqrt{D}(1+a|x|^c)^b}, \tag{8}$$

where a is a positive real number. From Eqs. (4), (6), and (7) we obtain

$$W(x,t) = C_1 \frac{[1+a(1-bc)|x|^c]\exp\left[-\frac{x^2}{4D\bar{t}(1+a|x|^c)^{2b}}\right]}{\sqrt{\bar{t}}(1+a|x|^c)^{1+b}}, \tag{9}$$

where $bc \leqslant 1$ in order to maintain $W(x,t)$ positive. For $a=0$ or $b=0$ we recover the Wiener process.

In particular, for $c=1/b$, we have

$$W(x,t) = C_2 \frac{\exp\left[-\frac{x^2}{4D\bar{t}(1+a|x|^{1/b})^{2b}}\right]}{\sqrt{\bar{t}}(1+a|x|^{1/b})^{1+b}}. \tag{10}$$

In this process, the behavior, for small a and x, is like a Gaussian function, whereas for large distances, the exponential term converges to a constant value. Therefore, for large distances, the dominant term is the multiplicative factor $1/(1+a|x|^{1/b})^{1+b}$, which approximates to the asymptotic power law $|x|^{-(1+b)/b}$. We note that, for instance, the power-lay decay is present in the fractional and nonlinear approaches [1,9].

For $b=1$ and $c=1/2$ we obtain

$$W(x,t) = C_3 \frac{[2+a\sqrt{|x|}]\exp\left[-\frac{x^2}{4D\bar{t}(1+a\sqrt{|x|})^2}\right]}{\sqrt{\bar{t}}(1+a\sqrt{|x|})^2}. \tag{11}$$

In this process, the behavior, for small a and x, is like a Gaussian function. For large distances, we basically have an exponential decay. We note that the exponential decay has been observed in pair dispersion in two-dimensional turbulence [10].

Moreover, for $bc<0$, the decay of the solution (9) is essentially a compressed Gaussian shape, whereas for $bc>0$, the decay is essentially a stretched Gaussian shape. It is interesting to emphasize that the solution (9) can have a similar asymptotic non-Gaussian shape of the random-walk model and time-fractional dynamic equation [1]. The asymptotic shape of the random-walk model and time-fractional dynamic equation is given by

$$W(x,t) \sim C_4 t^{-\alpha/2} \xi^{-(1-\alpha)/(2-\alpha)} \exp[-C_5 \xi^{2/(2-\alpha)}], \tag{12}$$

where $\xi \equiv |x|/t^{\alpha/2}$. This shape can be obtained from the solution (9), for large distances, by taking $bc=(1-\alpha)/(2-\alpha)$, $\bar{t}=t^{\alpha/(2-\alpha)}$ and $T^2(t)=[\alpha/(2-\alpha)]t^{2(\alpha-1)/(2-\alpha)}$.

Second case. We consider

$$D(x) = \sqrt{D}|x|^{-\theta/2}, \tag{13}$$

where θ is a real parameter. We should note that the diffusion coefficient (13) has been used to describe the diffusive process on a fractal [11]. The probability distribution (4) for the spatial multiplicative noise term (13) is given by

$$W(x,t) = \frac{|x|^{\theta/2}\exp\left[-\frac{|x|^{2+\theta}}{D(2+\theta)^2\bar{t}}\right]}{\sqrt{4\pi D\bar{t}}}. \tag{14}$$

In this process, we have the bimodal states. In fact, we can construct many bimodal states by choosing different functions for $D(x)$. For $\theta=0$ we recover the Wiener process. Basically, for large distances, the probability distribution (14) has a non-Gaussian decay. The second moment related to this process is given by

$$\langle x^2 \rangle = \frac{[D^2(2+\theta)^4]^{1/(2+\theta)}\Gamma\left[\frac{6+\theta}{2(2+\theta)}\right]\bar{t}^{\,2/(2+\theta)}}{\sqrt{\pi}}. \tag{15}$$

The solution (14) also reproduces the asymptotic shape (12) by taking $2+\theta=2/(2-\alpha)$, $\bar{t}=t^{\alpha/(2-\alpha)}$ and $T^2(t)=[\alpha/(2-\alpha)]t^{2(\alpha-1)/(2-\alpha)}$. The second moment (15) yields $\langle x^2\rangle \sim t^\alpha$, which corresponds to the same behavior of the time-fractional diffusion equation [1]. For this process, the multiplicative noise term corresponds to $g(x,t) \sim |x|^a t^b$, which has the same form suggested by Hentschel and Procaccia to study the turbulent system [8]. The asymptotic shapes above are summarized in Table I.

We can now compare with the solution obtained by the Ito approach for the same Langevin equation using (13). The solution has been obtained in [12], and for $T(t)=1$ it is given by

$$W_I(x,t) \sim \frac{|x|^{\theta}\exp\left[-\dfrac{|x|^{2+\theta}}{D(2+\theta)^2 t}\right]}{t^{(1+\theta)/(2+\theta)}}. \tag{16}$$

The second moment yields

$$\langle x^2\rangle_I \sim t^{2/(2+\theta)}. \tag{17}$$

We see that these two approaches give different behaviors for the probability distribution due to the multiplicative factors. However, the second moments of these two approaches give the same behavior. Further, both the distributions present the bimodal states. It is also interesting to compare them with another approach that uses the postpoint discretization rule [12,13]. For this last case, the probability distribution does not present the bimodal states; however, its second moment has the same power-law behavior of the Ito and Stratonovich approaches. We see that three different approaches give different behaviors, but they give the same power-law behavior for the second moment.

In summary, we have investigated the usual one-dimensional Fokker-Planck equation with a variable diffusion coefficient in the Stratonovich approach. We have considered a very general class of the multiplicative noise term, $g(x,t)=D(x)T(t)$, and we have presented the formal solution for the probability distribution. Using the formal solution, we have analyzed some particular solutions by choosing simple functions for $D(x)$. We have shown interesting behaviors for the probability distribution such as non-Gaussian, exponential, and power-law decays for large distances. As we note, the usual Fokker-Planck equation can describe many different anomalous processes with many different behaviors. The introduction of a time-dependent multiplicative noise term may be necessary for the cases of more complex systems such as turbulent systems, as suggested by several authors [7,8]. In fact, we have shown that the asymptotic shape of the random-walk model and time-fractional dynamic equation can be obtained from the solutions described in this paper with the time-dependent multiplicative noise term. The power-law decay generated from other approaches can also be reproduced from the solution (9). These results show that the asymptotic shapes of several other approaches can be generated from a single and well-established Langevin equation, by employing simple functions for the multiplicative noise term. However, it does not mean that our solutions can be used to substitute the solutions of other approaches due to the fact that their formulations are different, and this can also be viewed from our solutions, which exhibit different dynamics for small x in comparison with the results of other approaches. Further, we have also shown that the solutions of the Ito, Stratonovich, and postpoint discretization approaches, for $D(x)=\sqrt{D}|x|^{-\theta/2}$, describe different behaviors, but their second moments describe the same behavior. If the diffusion coefficient $D(x)=\sqrt{D}|x|^{-\theta/2}$ may describe exactly a real physical system by the Fokker-Planck equation, then further information of the microscopic structure of the system is necessary in order to choose which of the above approaches is the correct one or, simply, which of the above approaches can fit the experimental data.

[1] R. Metzer and J. Klafter, Phys. Rep. **339**, 1 (2000); J. Phys. A: Math. Gen. **37**, R161 (2004).

[2] R. Muralidhar, D. Ramkrishna, H. Nakanishi, and D. Jacobs, Physica A **167**, 539 (1990); K. G. Wang, L. K. Dong, X. F. Wu, and T. Ko, *ibid.* **203**, 53 (1994); K. G. Wang and M. Tokuyama, *ibid.* **265**, 341 (1999).

[3] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1996); C. W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1997).

[4] C. Tsallis, S. V. F. Levy, A. M. C. Souza, and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995); C. Tsallis and D. J. Bukman, Phys. Rev. E **54**, R2197 (1996); D. H. Zanette and P. A. Alemany, Phys. Rev. Lett. **75**, 366 (1995).

[5] V. M. Kenkre, E. W. Montroll and M. F. Shlesinger, J. Stat. Phys. **9**, 45 (1973); V. M. Kenkre, Phys. Lett. **65A**, 391 (1978); *Stochastic Processes in Chemical Physics: The Master Equation*, edited by I. Oppenheim, K. E. Shuler, and G. H. Weiss, (MIT Press, Cambridge, MA, 1977).

[6] J. Klafter, M. F. Shlesinger, and G. Zumofen, Phys. Today **49**(2), 33 (1996); H. Scher and E. W. Montroll, Phys. Rev. B **12**, 2455 (1975); G. Pfister and H. Scher, *ibid.* **15**, 2062 (1977); Adv. Phys. **27**, 747 (1978); G. Zumofen, A. Blumen, and J. Klafter, Phys. Rev. A **41**, R4558 (1990); P. W. M. Blom and M. C. J. M. Vissenberg, Phys. Rev. Lett. **80**, 3819 (1988).

[7] L. F. Richardson, Proc. R. Soc. London, Ser. A **110**, 709 (1926); A. N. Kolmogorov, C. R. (Dokl.) Acad. Sci. URSS **30**, 301 (1941); G. K. Batchelor, Proc. Cambridge Philos. Soc. **48**, 345 (1952); A. Okubo, J. Oceanogr. Soc. Jpn. **20**, 286 (1962).

[8] H. G. E. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984).

[9] I. T. Pedron, R. S. Mendes, L. C. Malacarne, and E. K. Lenzi, Phys. Rev. E **65**, 041108 (2002).

[10] M. C. Jullien, J. Paret, and P. Tabeling, Phys. Rev. Lett. **82**, 2872 (1999).

[11] B. O'Shaughnessy and I. Procaccia, Phys. Rev. Lett. **54**, 455 (1985); R. Balescu, *Statistical Dynamics: Matter Out of Equilibrium* (Imperial College Press, Singapore, 1997), Chap. 14.

[12] Kwok Sau Fa and E. K. Lenzi, Phys. Rev. E **67**, 061105 (2003); **71**, 012101 (2005).

[13] J. Dunkel and P. Hänggi, Phys. Rev. E **71**, 016124 (2005).